EFFECT OF UNIFORM DISPLACEMENT ON THE STRESS DISTRIBUTION OF A WOOD PLATE

By

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EFFECT OF UNIFORM DISPLACEMENT ON THE STRESS DISTRIBUTION OF A WOOD PLATE

I. Introduction

In a chapter devoted to strain-energy methods Timoshenko and Goodier[1] have explained the principle of virtual work. They define virtual displacement in the case of an elastic body as any small displacement compatible with the condition of continuity of the material and with the conditions for the displacements at the surface of the body, if such conditions are prescribed.

Let us consider the case of plane stress distribution in a plate. If from an orthotropic material a flat plate is cut parallel to a plane of elastic symmetry, it will have two perpendicular axes of symmetry in the plane of the plate. Such a plate is said to be orthotropic.

If we denote by u and v the components of actual displacements due to the load and by δu and δv the components of a virtual displacement from the loaded position of equilibrium, these latter components are arbitrary small quantities satisfying the conditions of continuity and the work done against the mutual actions between the particles is equal to the strain energy corresponding to the displacements.

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[1] S. Timoshenko and J. N. Goodier, Theory of Elasticity, Second Edition (New York, Toronto, and London: McGraw-Hill Book Company, 1951), Chapter on Strain-Energy Methods.

The principle of virtual work states that the total work done during the virtual displacement of any system in equilibrium must vanish. Hence we have:[2]

$$\delta\left\{\iint V_0\,dx\,dy-\iint(Xu+Yv)\,dx\,dy-\int(\bar{X}u+\bar{Y}v)\,ds\right\}=0 \qquad (1)$$

where δ means variation, X and Y are component body forces per unit volume of the plate, $\bar{X}$ and $\bar{Y}$ are components of body forces per unit area of the plate, and u and v are displacements in the x and y-directions, and where the first term in the bracket is the potential energy of deformation and the second and third terms together represent the potential energy of forces acting on the body if the potential energy of such forces for the unstressed condition is taken as zero. The complete expression in the brackets of equation (1) represents the total potential energy of the system. We can then say that the total potential energy of the system at the position of equilibrium is a maximum or a minimum; and since, for stable equilibrium, it is always necessary to insist upon a positive work for any virtual displacement, the total potential energy of the system at this position is always a minimum.

For our problem we shall assume no body forces so that equation (1) becomes

$$\delta\left\{\iint V_0\,dx\,dy-\int(\bar{X}u+\bar{Y}v)\,ds\right\}=0 \qquad (2)$$

---

[2] Ibid., p. 1.

For a plate of unit thickness the strain energy per unit volume V_0 can be written as

$$V_0 = 1/2\,(X_x\, e_{xx} + Y_y\, e_{yy} + X_y\, e_{xy}) \qquad (3)$$

In general the components of boundary forces per unit area of the plate can be expressed as

$$\begin{aligned} \overline{X} &= lX_x + mX_y \\ \overline{Y} &= mY_y + lX_y \end{aligned} \qquad (4)$$

where l and m are the cosines of the angles between the normal and the axes x and y respectively.

For the problem which we wish to consider the boundary forces are to be exerted compressively normal to the boundaries $y = \pm b$ only and no shearing forces are applied.

Then $\overline{X} = 0$ and $\overline{Y} = Y_y$ (5)

For orthotropic material the components of stress and strain are connected by the following relations:[3]

$$\begin{aligned} e_{xx} &= a_{11} X_x + a_{12} Y_y \\ e_{yy} &= a_{21} X_x + a_{22} Y_y \\ e_{xy} &= a_{33} X_y \end{aligned} \qquad (6)$$

where $a_{11} = \dfrac{1}{E_x}$, $a_{12} = -\dfrac{\sigma_{yx}}{E_y}$, $a_{21} = -\dfrac{\sigma_{xy}}{E_x}$,

$$a_{22} = \frac{1}{E_y}, \quad a_{33} = \frac{1}{\mu_{xy}} \qquad (7)$$

---

[3] U. S. Forest Products Laboratory Report, No. 1503.

and for which E_x and E_y are Young's moduli in the x and y-directions, respectively. Poisson's ratio σ_{xy} is the ratio of the contraction parallel to the y-axis to the extension parallel to the x-axis associated with a tension parallel to the x-axis and similarly for σ_{yx}. The quantity μ_{xy} is the modulus of rigidity associated with the directions of x and y.

For an orthotropic plate like the one we shall use, the relation $\sigma_{xy} E_y = \sigma_{yx} E_x$ holds so that $a_{12} = a_{21}$ for our problem.[4]

Since the equations of equilibrium are obtained independently of the law connecting stresses and strains, they will be satisfied by a stress function ϕ such that

$$X_x = \frac{\partial^2 \phi}{\partial y^2}, \quad Y_y = \frac{\partial^2 \phi}{\partial x^2}, \quad X_y = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{8}$$

With the knowledge of equations (3), (5), (6), (7), and (8) we can write equation (2) for our problem in the form

$$\delta\left\{2\int_0^b\int_0^a\left[a_{22}\frac{\partial^2\phi}{\partial x^2} + a_{11}\frac{\partial^2\phi}{\partial y^2} + 2a_{12}\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} + a_{33}\frac{\partial^2\phi}{\partial x \partial y}\right]dx\,dy - 4\int_0^a (Y_y v)_{y=b}\,dx\right\} = 0 \tag{9}$$

If we denote the expression in brackets by V, we can write

$$\delta V = 0 \tag{10}$$

Then a stress function taken in the form

---

[4] See U. S. Forest Products Laboratory Report No. 1503, p. 7.

$$\phi = \phi_{11} C_{11} + \phi_{21} C_{21} + \phi_{12} C_{12} + \phi_{22} C_{22} + \cdots + \phi_{mn} C_{mn} + \cdots \qquad (11)$$

where C_{mn} for m = 1, 2, 3, . . . and n = 1, 2, 3, . . . can be determined from the minimum conditions

$$\frac{\partial V}{\partial C_{11}} = 0, \quad \frac{\partial V}{\partial C_{21}} = 0, \quad \frac{\partial V}{\partial C_{12}} = 0, \cdots, \frac{\partial V}{\partial C_{mn}} = 0, \cdots \qquad (12)$$

We shall apply this theory to the following problem.

II. The Stress Distribution of a Wood Plate with Uniform Displacement

Suppose we take a rectangular plate of Sitka spruce of unit thickness, length 2b, and width 2a. Referring to Figure 1, let us prevent the corners A, B, C, D from any horizontal displacement, and place such a compressive load distribution (to be determined later) along sides AB and CD, only, as to cause the vertical displacements of all points along AB to be -k and along CD to be +k, where k is a positive constant.

Stated formally, the boundary conditions which the stress function ϕ must satisfy are:

$$\begin{aligned} &\text{for } x = \pm a, \quad X_x = 0, \quad X_y = 0 \\ &\text{for } y = +b, \quad v = -k, \quad X_y = 0 \\ &\text{for } y = -b, \quad v = +k, \quad X_y = 0 \end{aligned} \qquad (13)$$

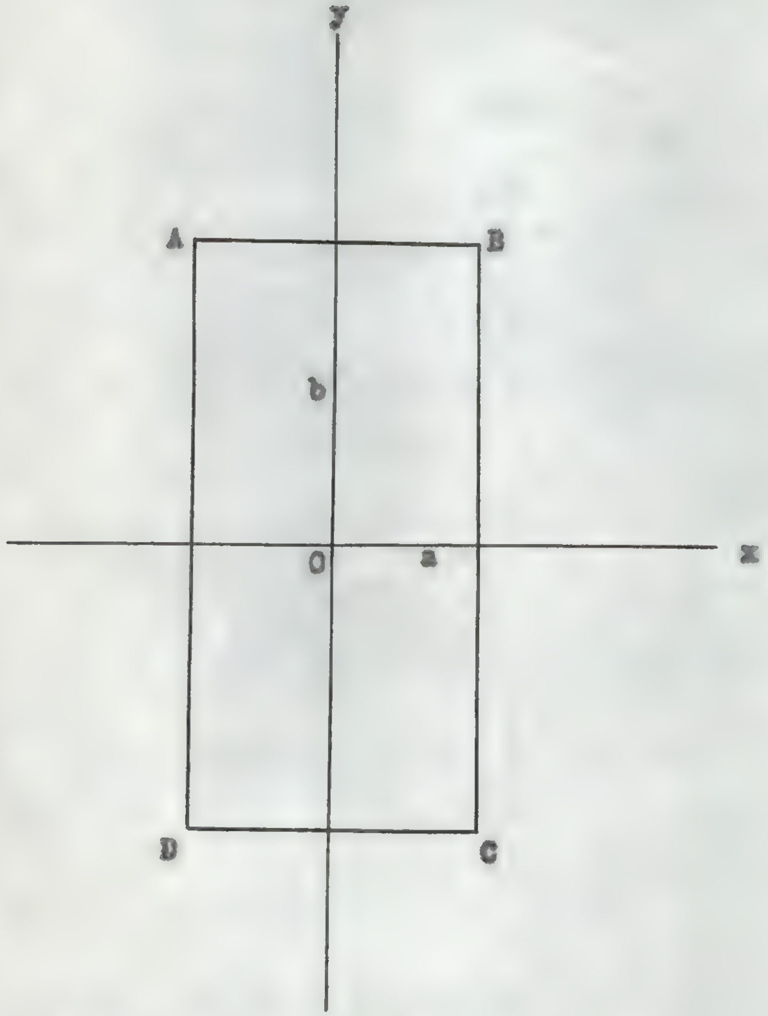

Figure 1. A rectangular plate of Sitka spruce of unit thickness.

at the corners (a,b), (a,-b), (-a,b), (-a,-b), $u = 0$

A suitable choice of the stress function ϕ for the problem under consideration is

$$\phi = A_k\left\{\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b} - \frac{a_{11}\pi^2}{2a_{12}b^2}x^2 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})C_{mn}\left[\left(\cos\frac{m\pi x}{a}-\cos m\pi\right)\cos\frac{n\pi y}{b} - \frac{a_{11}n^2\pi^2}{2a_{12}b^2}x^2\cos m\pi\cos n\pi\right]\right\} \tag{14}$$

where A_k (the k subscript on A_k indicates the number of constants C_{mn} that are included in each approximation) is an arbitrary constant to be adjusted to any desired total load on the sides $y = \pm b$ of the rectangle and δ_{mn} is the Kronecker delta.

Using equations (8) the stresses can readily be determined as follows:

$$X_y = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{A_k\pi^2}{ab}\left[\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})C_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right] \tag{15}$$

We observe that $X_y = 0$ along $x = \pm a$ in accord with the boundary conditions as given by equations (13)

$$X_x = \frac{\partial^2\phi}{\partial y^2} = -\frac{A_k\pi^2}{b^2}\left[\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})n^2C_{mn}\left(\cos\frac{m\pi x}{a}-\cos m\pi\right)\cos\frac{n\pi y}{b}\right] \tag{16}$$

We also see that $X_x = 0$ along $x = \pm a$ as should be true for the boundary conditions as given by equations (13).

$$
\begin{aligned}
Y_y = \frac{\partial^2 \phi}{\partial x^2} = -A_k \Big[& \frac{\pi^2}{a^2} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{a_{11}\pi^2}{a_{12} b^2} + \\
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1-\delta_{m1}\delta_{n1}) C_{mn} \Big(\frac{m^2\pi^2}{a^2} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} + \\
& \frac{a_{11} n^2 \pi^2 \cos m\pi \cos n\pi}{a_{12} b^2} \Big) \Big]
\end{aligned} \tag{17}
$$

Since $e_{xx} = \frac{\partial u}{\partial x}$, the first of equations (6) allows us to write

$$
\begin{aligned}
\frac{\partial u}{\partial x} = -\frac{A_k a_{11} \pi^2}{b^2} \Big[& \Big(1 + \cos\frac{\pi x}{a}\Big) \cos\frac{\pi y}{b} + \\
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1-\delta_{m1}\delta_{n1}) n^2 C_{mn} \Big(\cos\frac{m\pi x}{a} - \cos m\pi \Big) \cos\frac{n\pi y}{b} \Big] \\
- a_{12} \pi^2 \Big[& \frac{1}{a^2} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{a_{11}}{a_{12} b^2} + \\
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1-\delta_{m1}\delta_{n1}) C_{mn} \Big(\frac{m^2}{a^2} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} + \frac{a_{11} n^2 \cos m\pi \cos n\pi}{a_{12} b^2} \Big) \Big]
\end{aligned} \tag{18}
$$

and, integrating with respect to x,

$$
\begin{aligned}
u = -\frac{A_k a_{11} \pi^2}{b^2} \Big[& x \cos\frac{\pi y}{b} + \frac{a}{\pi} \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} + \\
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1-\delta_{m1}\delta_{n1}) n^2 C_{mn} \Big(\frac{a}{m\pi} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} - x \cos m\pi \cos\frac{n\pi y}{b} \Big) \Big] \\
- A_k a_{12} \pi^2 \Big[& \frac{1}{a\pi} \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{a_{11} x}{a_{12} b^2} + \\
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (1-\delta_{m1}\delta_{n1}) C_{mn} \Big(\frac{m}{a\pi} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} + \\
& \frac{a_{11} n^2 x \cos m\pi \cos n\pi}{a_{12} b^2} \Big) \Big]
\end{aligned} \tag{19}
$$

To check our boundary conditions again we see that $u = 0$ at the corners (a,b), (-a,b), (a,-b), (-a,-b).

Also, since $e_{yy} = \frac{\partial v}{\partial y}$, the second of equations (6) allows us to write

$$\frac{\partial V}{\partial y} = -\frac{A_k a_{12}\pi^2}{b^2}\Big[\Big(1+\cos\frac{\pi x}{a}\Big)\cos\frac{\pi y}{b} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})n^2 c_{mn}\Big(\cos\frac{m\pi x}{a}-\cos m\pi\Big)\cos\frac{n\pi y}{b}\Big]$$
$$-A_k a_{22}\pi^2\Big[\frac{1}{a^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} + \frac{a_{11}}{a_{12}b^2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})c_{mn}\Big(\frac{m^2}{a^2}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} + \frac{a_{11}n^2\cos m\pi\cos n\pi}{a_{12}b^2}\Big)\Big] \qquad (20)$$

and, integrating with respect to y, we have

$$V = -\frac{A_k a_{12}\pi^2}{b^2}\Big[\frac{b}{\pi}\Big(1+\cos\frac{\pi x}{a}\Big)\sin\frac{\pi y}{b} + \frac{b}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})n c_{mn}\Big(\cos\frac{m\pi x}{a}-\cos m\pi\Big)\sin\frac{n\pi y}{b}\Big]$$
$$-A_k a_{22}\pi^2\Big[\frac{b}{a^2\pi}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{a_{11}y}{a_{12}b^2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})c_{mn}\Big(\frac{m^2 b}{n\pi a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \frac{a_{11}n^2 y\cos m\pi\cos n\pi}{a_{12}b^2}\Big)\Big] \qquad (21)$$

for which

$$V_{y=\pm b} = \mp\frac{A_k a_{11}a_{22}\pi^2}{a_{12}b}\Big[1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})n^2 c_{mn}\cos m\pi\cos n\pi\Big] \qquad (22)$$

$\equiv$ a constant.

Substituting from equations (15), (16), (17), (22) into equation (10) gives

$$\delta V = 2\,(A_k)^2\int_0^b\!\!\int_0^a\Big\{a_{22}\Big[\frac{\pi^2}{a^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}+\frac{a_{11}\pi^2}{a_{12}b^2}+$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})C_{mn}\Big(\frac{m^2\pi^2}{a^2}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}+$$

$$\frac{a_{11}n^2\pi^2\cos m\pi\cos n\pi}{a_{12}b^2}\Big)\Big]^2+\frac{a_{11}\pi^4}{b^4}\Big[\Big(1+\cos\frac{\pi x}{a}\Big)\cos\frac{\pi y}{b}+$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})n^2C_{mn}\Big(\cos\frac{m\pi x}{a}-\cos m\pi\Big)\cos\frac{n\pi y}{b}\Big]^2$$

$$+\frac{2a_{12}\pi^2}{b^2}\Big[\frac{\pi^2}{a^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}+\frac{a_{11}\pi^2}{a_{12}b^2}+ \qquad (23)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})C_{mn}\Big(\frac{m^2\pi^2}{a^2}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}+\frac{a_{11}n^2\pi^2\cos m\pi\cos n\pi}{a_{12}b^2}\Big)\Big]\cdot$$

$$\Big[\Big(1+\cos\frac{\pi x}{a}\Big)\cos\frac{\pi y}{b}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})n^2C_{mn}\Big(\cos\frac{m\pi x}{a}-$$

$$\cos m\pi\Big)\cos\frac{n\pi y}{b}\Big]+\frac{a_{33}\pi^2}{a^2b^2}\Big[\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}+$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})C_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\Big]\Big\}\,dx\,dy-$$

$$\frac{4(A_k)^2aa_{11}^2a_{22}\pi^4}{a_{12}^2b^3}\Big[1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{m1}\delta_{n1})n^2C_{mn}\cos m\pi\cos n\pi\Big]^2=0$$

We know that the more terms that are used for the stress function ϕ the greater the accuracy of our results. However, experience with past problems based upon the principle of virtual displacement has usually shown that the use of about three constants yields results of a high degree of accuracy. We shall limit our applied problem to the use of three constants in our discussions.

If, as a first approximation, we take

$$\phi = A_1\Big\{\Big(1+\cos\frac{\pi x}{a}\Big)\cos\frac{\pi y}{b}-\frac{a_{11}\pi^2}{2a_{12}b^2}x^2+C_{21}\Big[\Big(\cos\frac{2\pi x}{a}-1\Big)\cos\frac{\pi y}{b}+$$

$$\frac{a_{11}\pi^2x^2}{2a_{12}b^2}\Big]\Big\}, \qquad (23)$$

substitute into equation (22), find $\frac{\partial V}{\partial c_{21}}$ and set $\frac{\partial V}{\partial c_{21}}$ = 0, we have the relation

$$16 a_{22} r^4 a_{12}^2 - 4a_{11}^2 a_{22} + 3a_{11} a_{12}^2 + 8a_{12}^3 r^2 + 4a_{33} r^2 a_{12}^2) c_{21} = 2a_{11} (a_{12}^2 - 2a_{11} a_{22}) \tag{24}$$

where $r = \frac{b}{a}$.

Taking as a second approximation

$$\phi = A_2 \left\{ \left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b} - \frac{a_{11}\pi^2}{2a_{12}b^2}x^2 + c_{21}\left[\left(\cos\frac{2\pi x}{a}-1\right)\cos\frac{\pi y}{b} + \frac{a_{11}\pi^2 x^2}{2a_{12}b^2}\right] + c_{12}\left[\left(\cos\frac{\pi x}{a}+1\right)\cos\frac{2\pi y}{b} + \frac{2a_{11}\pi^2 x^2}{a_{12}b^2}\right]\right\} \tag{25}$$

$$\frac{\partial V}{\partial c_{21}} = 0 \qquad \frac{\partial V}{\partial c_{12}} = 0$$

we get

$$c_{21}(16r^4 a_{22} a_{12}^2 - 4a_{11}^2 a_{22} + 3a_{11} a_{12}^2 + 8r^2 a_{12}^3 + 4r^2 a_{33} a_{12}^2) + c_{12}(-16a_{11}^2 a_{22}) = 2a_{11}(a_{12}^2 - 2a_{11}a_{22})$$

(26)

$$c_{21}(-16a_{11}^2 a_{22}) + c_{12}(a_{22} r^4 a_{12}^2 - 64a_{11}^2 a_{22} + 48a_{11}a_{12}^2 + 8a_{12}^3 r^2 + 4a_{33} r^2 a_{12}^2) = -16a_{11}^2 a_{22}$$

Similarly, as a third approximation, using the three terms involving the coefficients C_{12}, C_{21}, and C_{22} in equation (13) we have

$$
\begin{aligned}
&C_{21}\left(16a_{22}r^4a_{12}^2-4a_{11}^2a_{22}+3a_{11}a_{12}^2+8a_{12}^3r^2+4a_{33}r^2a_{12}^2\right)\\
&+C_{12}\left(-16a_{11}^2a_{22}\right)+C_{22}\left(16a_{11}^2a_{22}\right)=-4a_{11}^2a_{22}+2a_{11}a_{12}^2\\[1em]
&C_{21}\left(-16a_{11}^2a_{22}\right)+C_{12}\left(a_{12}^2a_{22}r^4-64a_{11}^2a_{22}+48a_{11}a_{12}^2\right.\\
&\left.+8r^2a_{12}^3+4a_{33}r^2a_{12}^2\right)+C_{22}\left(64a_{11}^2a_{22}-32a_{11}a_{12}^2\right)=\\
&\qquad\qquad -16a_{11}^2a_{22} \qquad\qquad (27)\\[1em]
&C_{21}\left(16a_{11}^2a_{22}\right)+C_{12}\left(64a_{11}^2a_{22}-32a_{11}a_{12}^2\right)+\\
&C_{22}\left(16r^4a_{12}^2a_{22}-64a_{11}^2a_{22}+48a_{11}a_{12}^2+32r^2a_{12}^3+\right.\\
&\left.16r^2a_{12}^2a_{33}\right)=16a_{11}^2a_{22}
\end{aligned}
$$

To illustrate the problem for a particular case, suppose we take a plain-sawn plate of Sitka spruce with the grain of the wood parallel to the y-axis for which the elastic moduli are taken to be those given by Table I, which follows.

TABLE I

ELASTIC MODULI OF SITKA SPRUCE*

σ_{yx}	0.464
E_x	0.076×10^6 lbs. per sq. in.
E_y	1.679×10^6 lbs. per sq. in.
μ_{xy}	0.112×10^6 lbs. per sq. in.

*The elastic moduli of Sitka spruce are average values obtained from a large number of tests made at the U. S. Forest Products Laboratory.

Due to equations (7) the values given in Table I lead to the values given in Table II for $a_{11}, a_{12} = a_{21}, a_{22}, a_{33}$.

TABLE II

ELASTIC COEFFICIENTS FOR SITKA SPRUCE

a_{11}	13.157×10^{-6} lbs. per sq. in.
a_{22}	0.59559×10^{-6} lbs. per sq. in.
a_{12}	$-\ 0.27635 \times 10^{-6}$ lbs. per sq. in.
a_{21}	$-\ 0.27635 \times 10^{-6}$ lbs. per sq. in.
a_{33}	8.9286×10^{-6} lbs. per sq. in.

Using the values of a_{mn} given by Table II and $r = 4$ in equations (24), (26), and (27) we arrive at the values of C_{mn} given in Table III, which follows.

TABLE III

VALUES OF C_{mn} FOR SUCCESSIVE APPROXIMATIONS FOR SITKA SPRUCE

	C_{21}	C_{12}	C_{22}
First approximation	+ 2.25400		
Second approximation	+ 0.03646	+ 0.24470	
Third approximation	+ 0.03359	+ 0.20620	- 0.03887

Suppose we have a compression whose total load is P, where P is any positive constant.

Considering the case for the third approximation, we obtain the total load required by the formula

$$2\int_0^a (Y_y)_{y=\pm b}\,dx = \frac{2a\,a_{11}\pi^2}{a_{12}\,b^2}(4c_{12}+c_{21}-4c_{22}-1)A_3 = -P \tag{28}$$

For our applied problem we must therefore take

$$A_3 = +1.227P \tag{29}$$

We are very much interested in the displacements u and v along $x=\pm a$ and $y=\pm b$ respectively.

Letting $x=\pm a=\pm 1$ in equation (19) for the case where $k=3$ and substituting from equation (29) into equation (19) we have

$$u_{x=\pm a} = \pm\left(-9.627\cos\frac{\pi y}{b} - 9.765\cos\frac{2\pi y}{b} + .1382\right)\times 10^{-6}P \tag{30}$$

The results for displacement u are shown in Table IV and by a solid line in Figure 2a for the third approximation.

TABLE IV

DISPLACEMENT u ALONG $x = \pm a$ (IN MULTIPLES OF 10^{-6} P UNITS)*

FOR THIRD APPROXIMATION	
y	$u_{x=\pm a}$
b	$\pm$ 0.000
7/8 b	$\pm$ 2.128
3/4 b	$\pm$ 6.945
5/8 b	$\pm$ 10.590
1/2 b	$\pm$ 9.909
3/8 b	$\pm$ 3.359
1/4 b	$\mp$ 6.669
1/8 b	$\mp$ 15.660
0	$\mp$ 19.250

*For rectangular plate of Sitka Spruce.

THE DISPLACEMENT u ALONG $x = a$ FOR SITKA SPRUCE AND AN ISOTROPIC MATERIAL IN A RECTANGULAR PLATE

Figure 2 b.
General shape of plate under compression.

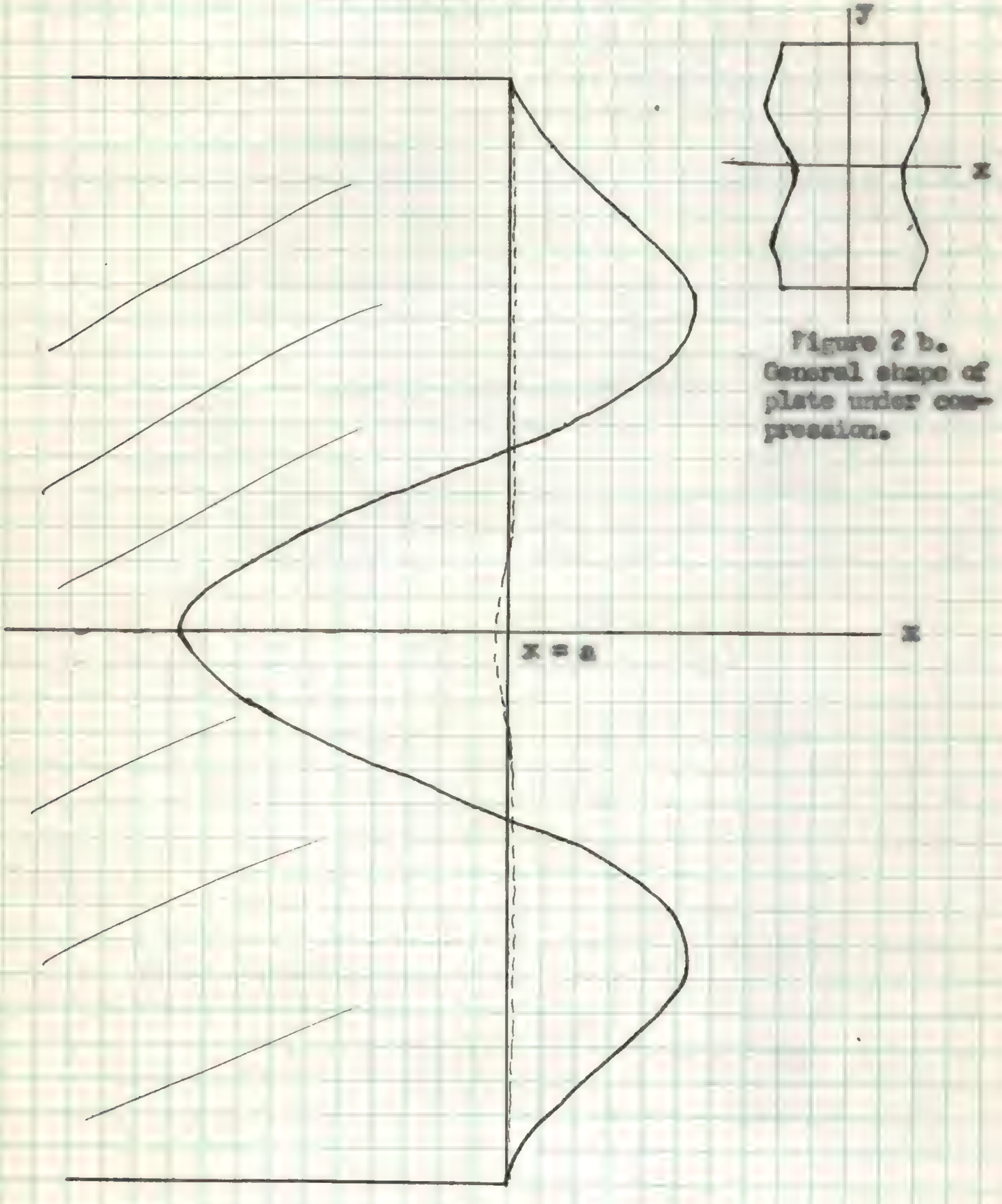

Figure 2 a.

Unit in y-direction - 1 space = 1/16 b
Unit in x-direction - 1 space = 2×10^{-6} P in. measured from $x = a$
Solid line - displacement u for Sitka Spruce
Dotted line - displacement u for isotropic material

For displacement v along $y=\pm b$ in the case where $k=3$ substitution of the constants into equation (22) gives

$$v_{y=\pm b} = \mp 1.191 \times 10^{-6} P \tag{31}$$

Now let us get the average value of u, call it $\bar{u}$, by

$$\bar{u} = \frac{2\int_0^b (u)_{x=a}\, dy}{2\int_0^b dy} \tag{32}$$

For the third approximation we find that

$$\bar{u} = 0.1382 \times 10^{-6} P \tag{33}$$

Let us see if

$$\frac{\frac{\bar{u}}{a}}{\frac{P}{2aE_y}} = \sigma_{yx} \tag{34}$$

We get

$$0.464 = 0.464 \tag{35}$$

Notice that μ has the value $\bar{\mu}$ at two points along the boundary $x=a$ where $y=.991b$ and at where $y=-.991b$.

Using three constants C_{21}, C_{12}, C_{22} from Table III, we have

$$(Y_y)_{y=\pm b} = \pm\pi^2(1.227P)\Big[(1-C_{12})\cos\frac{\pi x}{a} + 4(C_{21}-C_{22})\cos\frac{2\pi x}{a} + \frac{a_{11}}{a_{12}b^2}(4C_{12}+C_{21}-4C_{22}-1)\Big] \qquad (36)$$

$$(Y_y)_{y=\pm b} = \pm\left(9.613\cos\frac{\pi x}{a} + 3.509\cos\frac{2\pi x}{a} - .4999\right)P \qquad (37)$$

The values of Y_y along $y=\pm b$ as given by equation (37) are shown in Table V and in Figure 3 by a solid line.

TABLE V

Y_y ALONG $y = \pm b$ FOR EQUATION (37) (IN MULTIPLES OF P)*

FOR THIRD APPROXIMATION	
x	Y_y ALONG $y = \pm b$
a	$\mp$ 6.604
7/8 a	$\mp$ 6.900
3/4 a	$\mp$ 7.297
5/8 a	$\mp$ 6.660
1/2 a	$\mp$ 4.009
3/8 a	$\pm$ 0.696
1/4 a	$\pm$ 6.297
1/8 a	$\pm$ 10.860
0	$\pm$ 12.620

*For rectangular plate of Sitka Spruce.

LOAD DISTRIBUTION FOR SITKA SPRUCE AND ISOTROPIC MATERIAL IN A RECTANGULAR PLATE

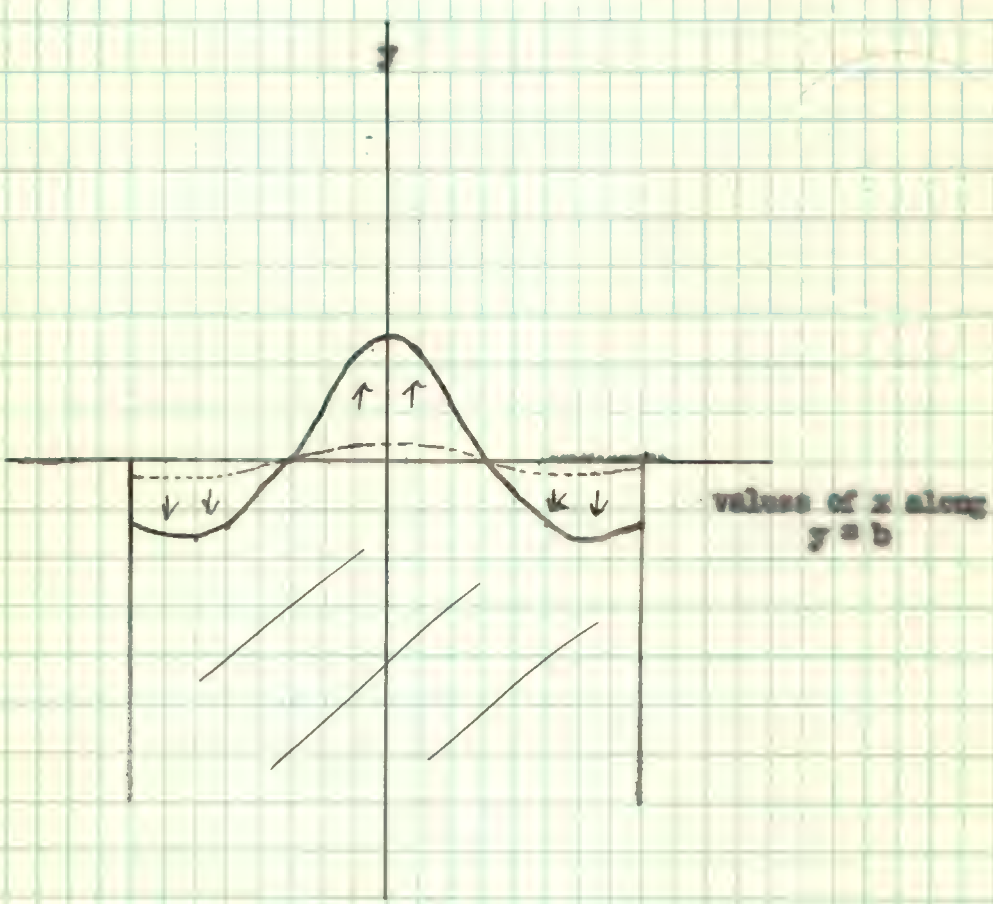

Figure 3

Unit along y-axis - 1 space = 5 P
Unit parallel to x-axis - 1 space = 1/8 a
Solid line - load distribution for Sitka Spruce
Dotted line - load distribution for isotropic material

In the case of a square plate (a = b) equations (27) become

$$
\begin{aligned}
-405.7\,C_{21} - 1650\,C_{12} + 1650\,C_{22} &= -410.4 \\
-1650\,C_{21} - 6547\,C_{12} + 6566\,C_{22} &= -1650 \\
1650\,C_{21} + 6566\,C_{12} - 6538\,C_{22} &= 1650
\end{aligned}
\qquad (38)
$$

Solving equations (38) simultaneously for C_{21}, C_{12}, and C_{22}, we have

$$
\begin{aligned}
C_{21} &= +0.51310 \\
C_{12} &= +0.07303 \\
C_{22} &= -0.04955
\end{aligned}
\qquad (39)
$$

Again, to maintain a total load of P, we have

$$
2\int_0^a (Y_y)_{y=\pm b}\,dx = \frac{2 a a_{11}\pi^2}{a_{12} b^2}\left(4C_{12} + C_{21} - 4C_{22} - 1\right)A_3 = -P \qquad (40)
$$

and substituting the values of C_{21}, C_{12}, C_{22} given by equations (40) and a = b = 1, we find that

$$
A_3 = +.3129\,P \qquad (41)
$$

Now looking at the displacements again, we see that

$$
u_{x=\pm a} = \mp 40.65\left[.4869\cos\frac{\pi y}{b} + .4904\cos\frac{2\pi y}{b} - .00304\right]\times 10^{-6}P \qquad (42)
$$

The values of u along $x = \pm a$ are tabulated in Table VI, which follows, and are plotted in Figure 4 by a solid line.

TABLE VI

VALUES OF DISPLACEMENT u ALONG $x = \pm a$ (IN MULTIPLES OF 10^{-6} P) *

y	$u_{x = \pm a}$
b	0.0000
7/8 b	$\pm$ 4.3140
3/4 b	$\pm$ 14.1200
5/8 b	$\pm$ 21.7900
1/2 b	$\pm$ 20.0000
3/8 b	$\pm$ 6.6460
1/4 b	$\mp$ 13.8700
1/8 b	$\mp$ 32.2600
0	$\mp$ 39.6100

---

*For a square plate of Sitka Spruce.

THE DISPLACEMENT u ALONG $x=a$ FOR SITKA SPRUCE AND ISOTROPIC MATERIAL IN A SQUARE PLATE

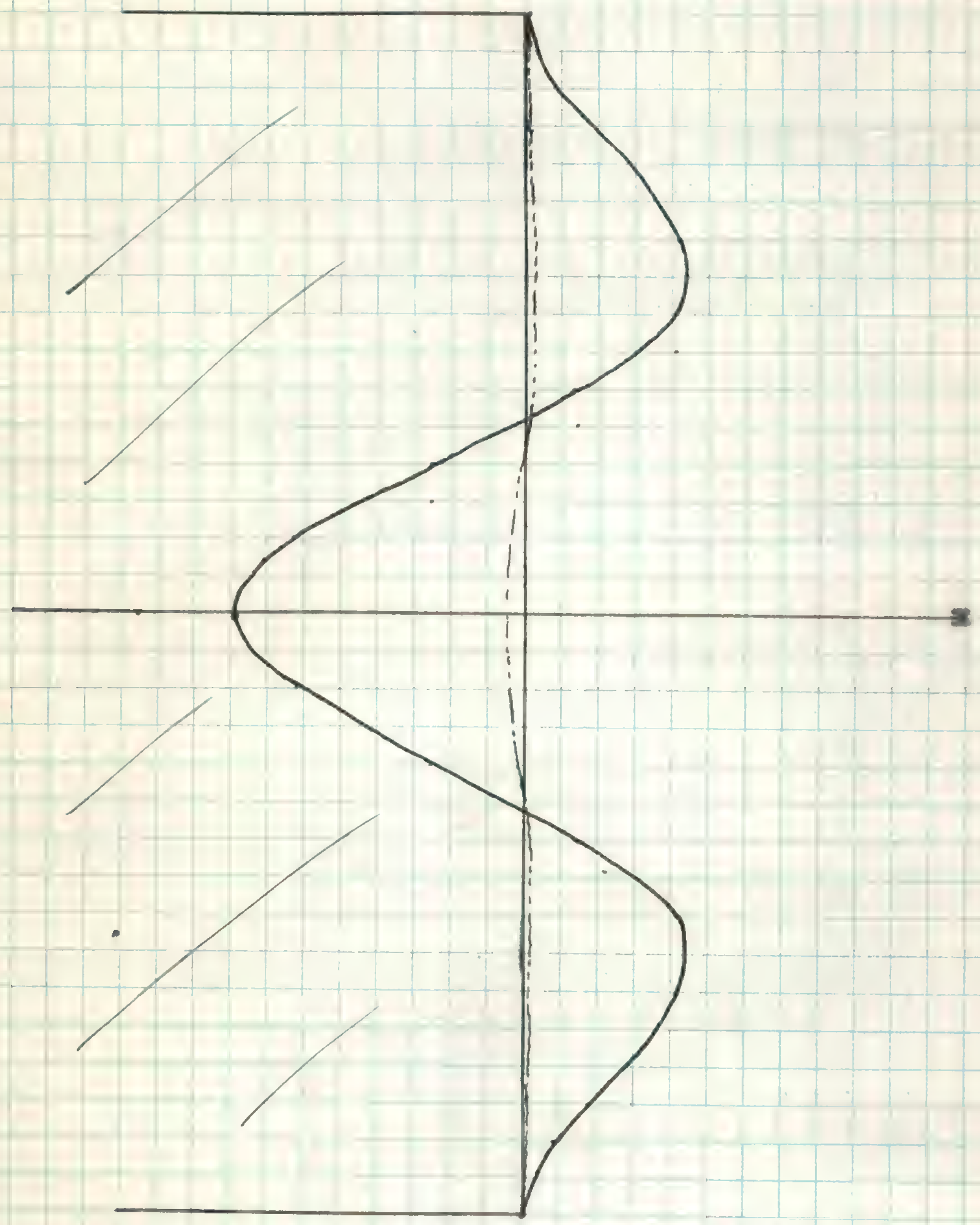

Figure 4

Unit in y-direction - 1 space = 1/16 b
Unit in x-direction - 1 space = 5×10^{-6} P in. measured from x = a
Solid line - displacement u for Sitka Spruce
Dotted line - displacement u for Isotropic Material

The displacement V along $y = \pm b$ turns out to be

$$V_{y=\pm b} = \mp .2952 \times 10^{-6} P \tag{43}$$

The formula showing the nature of the load distribution

$$(Y_y)_{y=\pm b} = \pi^2 A_3 \Big[(1 - C_{12}) \cos\frac{\pi x}{a} + 4(C_{21} - C_{22}) \cos\frac{2\pi x}{a} + \frac{a_{11}}{a_{12} b^2} (4C_{12} + C_{21} - 4C_{22} - 1) \Big] \tag{44}$$

turns out to give

$$(Y_y)_{y=\pm b} = \pm \Big(1.504 \cos\frac{\pi x}{a} + 1.514 \cos\frac{2\pi x}{a} - .1448 \Big] P \tag{45}$$

These values are tabulated in Table VII, which follows, and are plotted in Figure 5 by a solid line.

TABLE VII

VALUES OF V_y ALONG $y = \pm b$ (IN MULTIPLES OF P)*

x	$(V_y)_{y=\pm b}$
a	$\mp$ 0.1368
7/8 a	$\mp$ 0.4638
3/4 a	$\mp$ 1.2080
5/8 a	$\mp$ 1.7920
1/2 a	$\mp$ 1.6990
3/8 a	$\mp$ 0.6397
1/4 a	$\pm$ 0.9186
1/8 a	$\pm$ 2.3150
0	$\pm$ 2.8730

*For a square plate of Sitka Spruce.

LOAD DISTRIBUTION FOR SITKA SPRUCE AND ISOTROPIC MATERIAL IN A SQUARE PLATE

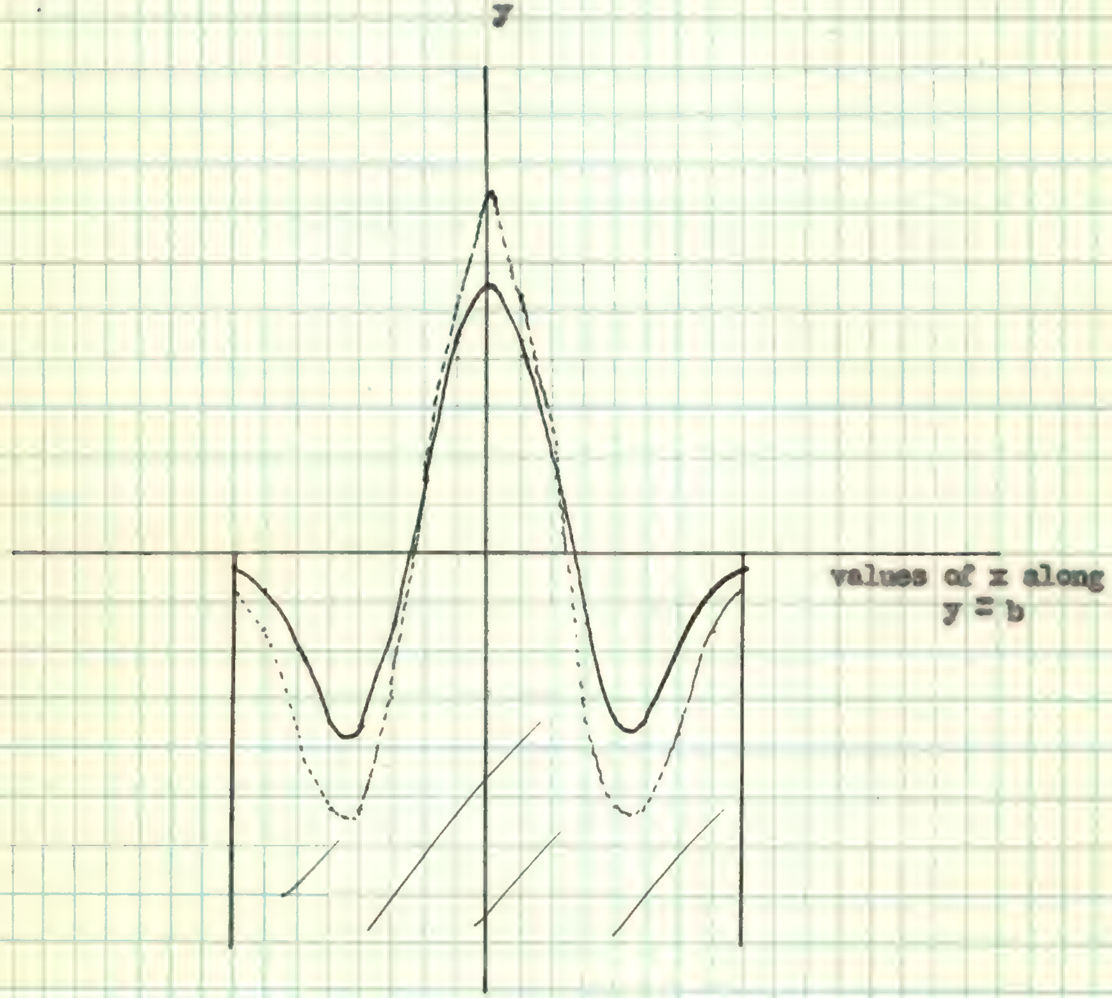

Figure 5

Unit along y - axis - 1 space = .5 P
Unit parallel to x - axis - 1 space = 1/8 a
Solid line - load distribution for Sitka Spruce
Dotted line - load distribution for isotropic material

Comparison of the results for the square plate with those for the rectangular plate shows that the results, in a general way, are the same.

This shows us that the shape of the boundaries along $x = \pm a$ taken after displacement under compression can be depended upon to be of the same general type regardless of the relative length and width of the plate.

III. The Isotropic Stress Function As A Special Case of the Orthotropic Stress Function.

For isotropic material

$$E_x = E_y = E, \quad \sigma_{yx} = \sigma, \quad \mu_{xy} = \frac{E}{2(1+\sigma)} \tag{46}$$

so that

$$a_{11} = \frac{1}{E}, \; a_{12} = a_{21} = -\frac{\sigma}{E}, \; a_{22} = \frac{1}{E}, \; a_{33} = \frac{2(1+\sigma)}{E}$$

Structural steel is a good example of isotropic material. Suppose we consider a plate of such steel having the same dimensions and subject to the same boundary conditions as we had for the wood plate and compare the displacement u along $x = \pm a$ for an isotropic material such as structural steel with the displacement u along $x = \pm a$ for an orthotropic material such as Sitka spruce.

Substitution from equations (46) into equations (27), using $\sigma = .3$ for structural steel, $r = \frac{b}{a} = 4$, gives

$$\begin{aligned} 37643C_{21} - 1600C_{12} + 1600C_{22} &= -382 \\ -1600C_{21} - 2512C_{12} + 6112C_{22} &= -1600 \\ 1600C_{21} + 6112C_{12} + 35504C_{22} &= 1600 \end{aligned} \tag{47}$$

which, when solved simultaneously, show that

$$\begin{aligned} C_{21} &= +0.01381 \\ C_{12} &= +0.51900 \\ C_{22} &= -0.04490 \end{aligned} \tag{48}$$

To maintain a total load of compression P we take

$$2\int_0^a \left(Y_y\right)_{y=\pm b} dx = -\frac{2a\pi^2}{\sigma b^2}\left(4C_{12}+C_{21}-4C_{22}-1\right)A_3' = -P \tag{49}$$

$$A_3' = +0.1915P \tag{50}$$

where the prime on the A_3 signifies that we are dealing with the isotropic case.

The displacement

$$u_{x=\pm a} = \pm\left\{ -\frac{a\pi^2 A_3'}{E b^2} \left[(1-C_{21})\cos\frac{\pi y}{b} + 4(C_{12}-C_{22})\cos\frac{2\pi y}{b} + (1-4C_{12}-C_{21}+4C_{22}) \right] \right\} \tag{51}$$

where, for structural steel, E = 30 x 10^6 lbs. per square inch

$$u_{x=\pm a} = \pm\left(-.003884\cos\frac{\pi y}{b} - .008882\cos\frac{2\pi y}{b} + .004998\right)10^{-6}P \tag{52}$$

Table VIII, which follows, shows the values of $u_{x=\pm a}$ in multiples of 10^{-6} P.

TABLE VIII

VALUES OF $\mu_{x=\pm a}$ (IN MULTIPLES OF 10^{-6} P)*

y	$\mu_{x=\pm a}$
b	$\pm$ 0.000000
7/8 b	$\pm$ 0.002306
3/4 b	$\pm$ 0.007744
5/8 b	$\pm$ 0.012760
1/2 b	$\pm$ 0.013880
3/8 b	$\pm$ 0.009792
1/4 b	$\pm$ 0.002252
1/8 b	$\mp$ 0.004870
0	$\mp$ 0.007768

*For rectangular plate, isotropic material.

The values of $u_{x=\pm a}$ given in Table VIII are plotted in Figure 2a as shown by a dotted line. Figure 2b shows in greatly exaggerated form the general shape of the plate during compression.

They show that the general shape of the curve representing the displacement u in the x-direction is the same for both materials but that the displacements are very much smaller for the isotropic material.

The displacement v along $y = \pm b$ is

$$v_{y=\pm b} = \mp \frac{\pi^2 A_3'}{E\delta b}(4C_{12}+C_{21}-4C_{22}-1) \qquad (53)$$

with substitutions from equations (46) and (48) giving

$$v_{y=\pm b} = \mp 0.2677 \times 10^{-6} p \qquad (54)$$

This value is smaller than for the orthotropic material, as should be expected for the same load P.

The stress distribution along the top and bottom of the plate is given by

$$(Y_y)_{y=\pm b} = \pm A_3'\pi^2\Big[(1-C_{12})\cos\frac{\pi x}{a} + 4(C_{21}-C_{22})\cos\frac{2\pi x}{a} - \frac{1}{\delta b^2}(4C_{12}+C_{21}-4C_{22}-1)\Big] \qquad (55)$$

Substituting the values of C_{mn} from equations (43) and the values of a_{mn} from equations (46) into equation (55) we have

$$(Y_y)_{y=\pm b} = \pm(.9091\cos\frac{\pi x}{a} + .4430\cos\frac{2\pi x}{a} - .2645)P \tag{56}$$

Table IX, which follows, gives the values of Y_y along $y=\pm b$ in multiples of P. The values of Y_y along $y=\pm b$ given by Table IX are plotted in Figure 3 as shown by a dotted line.

TABLE IX

VALUES OF Y_y ALONG $y = \pm b$ (IN MULTIPLES OF P) *

x	$(Y_y)_{y=\pm b}$
a	$\mp$ 0.7306
7/8 a	$\mp$ 0.7912
3/4 a	$\mp$ 0.9073
5/8 a	$\mp$ 0.9256
1/2 a	$\mp$ 0.7075
3/8 a	$\mp$ 0.2298
1/4 a	$\pm$ 0.6915
1/8 a	$\pm$ 0.8886
0	$\pm$ 1.0876

*For rectangular plate, isotropic material.

In the case of a square plate (a = b) of isotropic material using the three constants C_{21} , C_{12} , C_{22} gives

$$\begin{aligned} -157\,C_{21} - 1600\,C_{12} + 1600\,C_{22} &= -382 \\ -1600\,C_{21} - 5887\,C_{12} + 6112\,C_{22} &= -1600 \\ 1600\,C_{21} + 6112\,C_{12} - 5536\,C_{22} &= 1600 \end{aligned} \qquad (57)$$

which, when solved simultaneously, gives

$$\begin{aligned} c_{21} &= +0.17660 \\ c_{12} &= +0.15920 \\ c_{22} &= -0.06219 \end{aligned} \qquad (58)$$

To maintain a total compression P we have

$$2\int_0^a \left(Y_y\right)_{y=\pm 1} dx = \frac{2\pi^2}{-b}\left(4C_{12}+C_{21}-4C_{22}-1\right)A_3' = -P$$

$$A_3' = +.2445\,P \qquad (59)$$

The displacement u along $x = \pm a$ is given by

$$u_{x=\pm a} = \mp \frac{\pi^2}{E} A_3' \Big[(1-C_{21}) \cos\frac{\pi y}{b} + 4(C_{12}-C_{22}) \cos\frac{2\pi y}{b}$$

$$+ (1-4C_{12}-C_{21}+4C_{22}) \Big] \qquad (60)$$

$$\mu_{x=\pm a} = \mp\left(.06623\cos\frac{\pi y}{b} + .0713\cos\frac{2\pi y}{b} - .005\right) \times 10^{-6} p$$

(61)

The values of μ along $x=\pm a$ are tabulated in Table X, which follows, and plotted in Figure 4 as shown by a dotted line.

TABLE I

VALUES OF u ALONG $x = \pm a$ (IN MULTIPLES OF 10^{-6} P) *

y	$u_{x=\pm a}$
b	0.0000
7/8 b	$\pm$ 0.0157
3/4 b	$\pm$ 0.0518
5/8 b	$\pm$ 0.0857
1/2 b	$\pm$ 0.0763
3/8 b	$\pm$ 0.0301
1/4 b	$\mp$ 0.0418
1/8 b	$\mp$ 0.1066
0	$\mp$ 0.1325

*For a square plate, isotropic material.

The displacement in the y-direction is given by

$$V_{y=\pm b} = \mp \frac{\pi^2 A_3'}{E b}(4C_{12}+C_{21}-4C_{22}-1) \tag{62}$$

$$V_{y=\pm b} = \mp 0.01666 \times 10^{-6} P \tag{63}$$

The values of V_y along $y=\pm b$ are obtained from the equation

$$(V_y)_{y=\pm b} = \pm\left(1.987\cos\frac{\pi x}{a} + 2.139\cos\frac{2\pi x}{a} - .5002\right)P \tag{64}$$

similar to previous procedure.

These values are tabulated in Table XI, which follows, and are plotted as shown in Figure 5 by a dotted line.

TABLE XI

VALUES OF V_y ALONG $y=\pm b$ (IN MULTIPLES OF P)*

x	$(V_y)_{y=\pm b}$
a	$\mp$ 0.3482
7/8 a	$\mp$ 0.8234
3/4 a	$\mp$ 1.9050
5/8 a	$\mp$ 2.7730
1/2 a	$\mp$ 2.6390
3/8 a	$\mp$ 1.2523
1/4 a	$\pm$ 0.9048
1/8 a	$\pm$ 2.8480
0	$\pm$ 3.6260

*For a square plate, isotropic material.

IV. Summary

Before summarizing we might mention the reason for setting up the boundary conditions in the manner set forth by equations (13). Due to friction between machine heads and block, boundary conditions such as we have used might very easily be largely realized in actual practice.

As has been true of so many problems in the past, one might have expected the displacement u along $x = \pm a$ to cause a simple bulging contour. Instead, however, we get the unexpected contour shown in Figure 2b.

It is very interesting to note that the average value of u, called $\bar{u}$, does not necessarily occur at large distances from the ends (as one might expect). In fact, in some cases, it appears that it occurs near the ends. This cautions against the common practice of placing the instruments for measuring Poisson's ratio near the middle.

We also see that in the case of isotropic material the unexpected contours still prevail but are barely perceptible.

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BIOGRAPHICAL SKETCH

The author was born on April 2, 1913 in Constantine, Michigan. Most of his pre-college years were spent in Colon, Michigan where he studied for seven years in preparation to be a concert pianist. He studied oil painting under a capable student of the Detroit Institute of Arts and was producing finished work at the age of 13. He was champion orator of his high school and received an award from The Detroit News. He was valedictorian of his class in the eighth grade and valedictorian of his class again in high school. As a result he was offered a scholarship at Western Michigan Teachers College, Kalamazoo, Michigan, which he succeeded in holding for all four years of his undergraduate training on the basis of his high scholastic record there and received his Bachelor of Arts degree from there in 1935. While there he was elected to membership in the Kappa Rho Sigma Honorary Society and invited to membership in Kappa Delta Pi.

Following graduation he entered the teaching profession and at the same time started his graduate work at the University of Michigan. He received the Master of Arts Degree in Education from the University of Michigan in 1938 and the Master of Arts degree in Mathematics from the University of Michigan in 1942. Since 1938 his teaching has been entirely on the college level. In 1949 he came to the University of Florida as a full time instructor and graduate student and continued on his graduate work to date.

This dissertation was prepared under the direction of the chairman of the candidate's Supervisory Committee and has been approved by all members of the committee. It was submitted to the graduate council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

R. E. Page
Dean

C. F. Byers

June, 1952

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